

CONVEX HULLS OF UNIFORM SAMPLES FROM A CONVEX POLYGON

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Abstract

In Groeneboom (1988) a central limit theorem for the number of vertices N_n of the convex hull of a uniform sample from the interior of convex polygon was derived. To be more precise, it was shown that $\{N_n - \frac{2}{3}r \log n\} / \{\frac{10}{27}r \log n\}^{1/2}$ converges in law to a standard normal distribution, if r is the number of vertices of the convex polygon from which the sample is taken.

In the unpublished preprint Nagaev and Khamdamov (1991) a central limit result for the joint distribution of N_n and A_n is given, where A_n is the area of the convex hull, using a coupling of the sample process near the border of the polygon with a Poisson point process as in Groeneboom (1988), and representing the remaining area in the Poisson approximation as a union of a doubly infinite sequence of independent standard exponential random variables. We derive this representation from the representation in Groeneboom (1988) and also prove the central limit result of Nagaev and Khamdamov (1991), using this representation. The relation between the variances of the asymptotic normal distributions of number of vertices and the area, established in Nagaev and Khamdamov (1991), corresponds to a relation between the actual sample variances of N_n and A_n in Buchta (2005). We show how these asymptotic results all follow from one simple guiding principle. This corrects at the same time the scaling constants in Cabo and Groeneboom (1994) and Nagaev (1995).

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1. Introduction

Let N_n be the number of vertices of the convex hull of a sample of size n , drawn uniformly from the interior of a convex polygon with r vertices. It was shown in Groeneboom (1988) that

$$\{N_n - \frac{2}{3}r \log n\} / \{\frac{10}{27}r \log n\}^{1/2} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

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Dedicated to the memory of Alexander Nagaev.

where $\mathcal{N}(0, 1)$ denotes the standard normal distribution. This was proved by coupling the sample point process near the boundary of the convex polygon with a Poisson point process, and showing that the relevant part of the sample process could be approximated sufficiently closely by the coupled Poisson point process. The central limit result for N_n was subsequently derived from a corresponding result for the boundary of the convex hull of the approximating Poisson point process. These methods were also applied to the area A_n of the convex hull in Cabo and Groeneboom (1994), but unfortunately the central limit result A_n contained a scaling error (see Remark 3.2).

Nagaev and Khamdamov (1991), using the coupling of (part of the) sample point process with a Poisson process introduced in Groeneboom (1988), derived the following interesting central limit theorem for the joint distribution of the number of vertices and the area of the convex hull of a uniform sample of n points on the interior of a convex polygon.

Theorem 1.1. (Theorem 1 of Nagaev and Khamdamov (1991)) *Let N_n denote the number of vertices of the convex hull of a uniform sample of size n from the interior of a convex polygon C with $r \geq 3$ vertices and area $A(C)$. Moreover, let A_n denote the area of the convex hull of the sample, and let the scaled “remaining area” \bar{A}_n be defined by*

$$\bar{A}_n = n \{A(C) - A_n\} / A(C)$$

Then

$$\left(\frac{10}{27}r \log n\right)^{-1/2} \left(N_n - \frac{2}{3}r \log n, \bar{A}_n - \frac{2}{3}r \log n\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma), \quad (1.1)$$

where $\mathcal{N}(0, \Sigma)$ denotes the normal distribution with expectation the zero vector and covariance matrix Σ given by

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & \frac{14}{5} \end{pmatrix}$$

This is an extension of the central limit theorem for the number of vertices N_n in Groeneboom (1988), and one indeed recovers the central limit theorem given there by specializing the above result to the first coordinate. Unfortunately, the preprint Nagaev and Khamdamov (1991), containing this result, was never published. Moreover, it is written in Russian and its length is 50 pages, which might also not have helped its spread in the scientific world.

In a private correspondence Christian Buchta revealed to me that the constant for the central limit theorem for the second component (the remaining area) in Nagaev and Khamdamov (1991) was consistent with a relation he had derived himself between the finite sample variances of N_n and \bar{A}_n .

It is the purpose of the present note to give a simple proof of Theorem 1.1, deriving the result from the central limit theorem for N_n in Groeneboom (1988). We think that using the central limit

theorem of Groeneboom (1988) considerably simplifies the proof of Theorem 1.1 in Nagaev and Khamdamov (1991) and perhaps more clearly reveals the beauty of their idea. The relation between the variances in Theorem 1.1 can be considered to be a precursor (in an asymptotic sense) of the relation found between the finite sample variances in Buchta (2005).

For recent work on central limit theorems for random polytopes, see, e.g., Bárány and Reitzner (2010a) and Bárány and Reitzner (2010b), where also references to earlier work in this area can be found.

2. Representation of the remaining area by i.i.d. exponentials

We consider the Poisson point process \mathcal{P} of intensity 1 in \mathbb{R}_+^2 , and its left-lower convex hull, as in Groeneboom (1988). To make the connection with Groeneboom (1988), we first restate the definition of the process of vertices $\{W(a) : a \in \mathbb{R}_+\}$ consisting of the vertices of the (left-lower) convex hull of a Poisson process \mathcal{P} with intensity 1 in \mathbb{R}_+^2 .

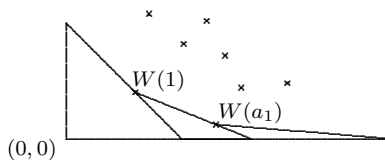


FIGURE 1: $W(a)$ -process

Definition 1. For each $a > 0$, $W(a) = (U(a), V(a))$ is the point of the realization of the Poisson process \mathcal{P} on \mathbb{R}_+^2 such that all points of the realization of \mathcal{P} lie to the right of the line of the line $x + ay = c$ which passes through $W(a)$. If there are several of such points (which happens with probability zero for fixed a), we define $U(a)$ ($V(a)$) as the supremum (infimum) of x -coordinates (y -coordinates) of points of this type.

We now have the following result (see also Theorem 2.1 of Nagaev (1995) for a result of this type).

Theorem 2.1. Let $a_0 = 1$, let a_1, a_2, \dots be the jump times of the process $\{W(a) : a \geq 1\}$, and let D_0 be the area of the isosceles triangle T_0 with a basis, running through $W(1)$, and two equal sides along the x - and y -axis, meeting at the top at the origin. Moreover, let D_i , $i \geq 1$, be the area of the triangle T_i , with top at $W(a_{i-1})$, basis along the x -axis, and sides along the lines $x + a_{i-1}y = U(a_i) + a_i V(a_i)$ and $x + a_i y = U(a_i) + a_i V(a_i)$, where $W(a_i)$, $U(a_i)$ and $V(a_i)$ are defined as in Definition 1. Then

- (i) The areas D_0, D_1, \dots form an i.i.d. sequence of standard exponential random variables.
- (ii) Let S_i be the length of the line segment, connecting $W(a_{i-1})$ and $W(a_i)$, and let L_i be the

length of the segment, obtained by extending the line segment from $W(a_{i-1})$ to $W(a_i)$ until it crosses the x -axis. Then the random variables S_i^2/L_i^2 , $i = 1, 2, \dots$ form an i.i.d. sequence of $\text{Uniform}(0, 1)$ random variables, independent of $W(1)$. Moreover, the S_i^2/L_i^2 are independent of the sequence D_0, D_1, \dots

Proof. (i). By Part (i) of Lemma 2.4 of Groeneboom (1988) we have, for $z \geq 0$,

$$P\{D_0 > z\} = P\left\{\frac{1}{2}\{U(1) + V(1)\}^2 > z\right\} = \int_{\{(x,y): \frac{1}{2}(x+y)^2 > z\}} e^{-\frac{1}{2}(x+y)^2} dx dy = e^{-z},$$

showing that D_0 has a standard exponential distribution. Let \mathcal{F}_a denote the σ -algebra, generated by the points $\{W(b), 1 \leq b \leq a\}$. Then, as shown in Groeneboom (1988), the process of points $\{W(a), a \geq 1\}$ is a Markov process w.r.t. the filtration $\{\mathcal{F}_a, a \geq 1\}$. Now note that, if $i \geq 1$, $D_i > z$ exactly when there are no points in the triangle of area z , with top at $W(a_i)$, basis along the x -axis, and sides along the lines $x + a_{i-1}y = U(a_i) + a_i V(a_i)$ and $x + a_i y = U(a_i) + a_i V(a_i)$. Since this event is independent of the location of the points $W(a_0), \dots, W(a_{i-1})$, by the Poisson property of the point process in \mathbb{R}_+^2 , we get:

$$P\{D_i > z\} = e^{-z}, \quad z \geq 0,$$

where the event $D_i > z$ is independent of D_0, \dots, D_{i-1} (note that we can use the strong Markov property here).

(ii). The jump measure $M(a, w; \cdot)$ of the process $\{W(a) : a > 0\}$ is given by

$$M(a, w; B) = \int_0^y u 1_B(au, -u) du, \quad (2.1)$$

see (2.22) of Groeneboom (1988). Hence, conditioning on $W(a) = W(a_{i-1}) = (x, y)$ and the event that there is a jump at time a , the location of the next vertex has a density proportional to u (representing the distance of $W(a)$ to the next vertex). So we get, for $z \in (0, 1)$,

$$\begin{aligned} & P\left\{S_i^2/L_i^2 < z \mid W(a) > W(a-) = (x, y)\right\} \\ &= P\left\{S_i < L_i \sqrt{z} \mid W(a) > W(a-) = (x, y)\right\} \\ &= P\left\{S_i < y \sqrt{z(1+a^2)} \mid W(a) > W(a-) = (x, y)\right\} \\ &= \frac{2}{y^2\{1+a^2\}} \int_0^{y\sqrt{z(1+a^2)}} u du = z, \end{aligned}$$

where we use that $\frac{1}{2}y^2\{1+a^2\}$ is the total measure of the jump measure on the line segment of length $y\sqrt{1+a^2}$, connecting (x, y) and $(x+ay, 0)$. This implies that S_i^2/L_i^2 has a uniform distribution, in accordance with Theorem 2.1 of Nagaev (1995). Moreover, since the distribution neither involves the value of $a = a_i$ nor that of $W(a_{i-1})$, the sequence of variables S_i^2/L_i^2 is i.i.d. For the same reason the

variable S_i^2/L_i^2 is independent of D_j , $j \leq i$. It is also seen that S_i^2/L_i^2 is independent of D_j , $j > i$, since the conditional distribution of D_{i+1} , given $W(a_i)$, is standard exponential, independently of the value of $W(a_i)$.

Corollary 1. *Let the sequences a_0, a_1, \dots and $V(a_0), V(a_1), \dots$ be defined as in Theorem 2.1, and let $\tau_i = V(a_i)/V(a_{i-1})$, $i = 1, 2, \dots$. Then the sequence of random variables τ_1, τ_2, \dots is i.i.d. and*

$$(1 - \tau_i)^2 \sim \text{Uniform}(0, 1).$$

Moreover, the random variables τ_i are independent of $V(a_0) = V(1)$ and the areas D_i , where D_i is defined as in Theorem 2.1.

Proof. This follows from part (ii) of Theorem 2.1 since

$$1 - \tau_i = 1 - \frac{V(a_i)}{V(a_{i-1})} = \frac{V(a_{i-1}) - V(a_i)}{V(a_{i-1})} = \frac{S_i}{L_i}, \quad i = 1, \dots,$$

where the last equality is the proportionality relation, well-known from elementary geometry.

The following result is the key to Theorem 1.1.

Corollary 2. *Let, for $m = 2, 3, \dots$, $N(1, m)$ be the number of jumps of the process $\{W(a) : a \in [1, m]\}$ and let $[EN(1, m)]$ be the largest integer smaller than or equal to $EN(1, m)$. Then:*

(i)

$$EN(1, m) = \frac{1}{3} \log m,$$

(ii) *As $m \rightarrow \infty$ the bivariate random variable*

$$\left(\{N(1, m) - EN(1, m)\} / \sqrt{\frac{5}{27} \log m}, \sum_{i=1}^{[EN(1, m)]} (D_i - 1) / \sqrt{EN(1, m)} \right)$$

converges in distribution to a bivariate normal distribution with expectation zero and covariance matrix equal to the identity matrix I .

Proof. (i). This is part (i) of Theorem 2.4 of Groeneboom (1988), which is a simple consequence of the fact that the expected jump rate of the process $\{W(a) : a \geq 1\}$ is given by $1/(3a)$.

(ii). The area D_i of the triangle T_i , as defined in Theorem 2.1, is given by:

$$D_i = \frac{1}{2} V(a_{i-1}) (V(a_{i-1}) + a_i V(a_{i-1}) - V(a_{i-1}) - a_{i-1} V(a_{i-1})) = \frac{1}{2} V(a_{i-1})^2 (a_i - a_{i-1}). \quad (2.2)$$

Define

$$U_i = U(a_i), \quad V_i = V(a_i), \quad \text{and} \quad W_i = (U_i, V_i), \quad i = 0, 1, \dots$$

It is clear that (2.2) gives a tridiagonal system for solving a_i in terms of the D_i and V_i . We get, using $a_0 = 1$,

$$a_n = 1 + 2 \sum_{i=1}^n \frac{D_i}{V_{i-1}^2}, \quad n \geq 1. \quad (2.3)$$

We now define, for $n \geq 1$,

$$Y_n = V_{n-1}^2 \left\{ 1 + 2 \sum_{i=1}^n \frac{D_i}{V_{i-1}^2} \right\} = V_{n-1}^2 a_n.$$

Thus,

$$\log a_n = -2 \log V_{n-1} + \log Y_n, \quad (2.4)$$

and hence we get the “switching relation”:

$$N(1, m) \geq n \iff a_n \leq m \iff -2 \log V_{n-1} + \log Y_n \leq \log m. \quad (2.5)$$

By Corollary 1:

$$EV_n^2 = EV_0^2 \prod_{i=1}^n \tau_i^2 = 6^{-n} EV_0^2, \quad E \left(\frac{V_n^2}{V_k^2} \right) = \prod_{i=k+1}^n E \tau_i^2 = 6^{-(n-k)}, \quad n > k \geq 0. \quad (2.6)$$

Since, by Theorem 2.1, the τ_i are also independent of the D_i , we obtain, for all $k \geq 1$,

$$\begin{aligned} EY_n &= 6^{-(n-1)} EV_0^2 + 2 \sum_{j=1}^n E \left(\frac{V_{n-1}^2}{V_{j-1}^2} \right) = 6^{-(n-1)} EV_0^2 + 2 \sum_{j=1}^{n-1} 6^{-j} \\ &\leq 6^{-(n-1)} EV_0^2 + 2 \sum_{j=1}^{\infty} 6^{-j}. \end{aligned}$$

This implies, by Markov's inequality,

$$Y_n = O_p(1), \quad n \rightarrow \infty.$$

Since we also have $Y_n \geq 2D_n$, for all $n \geq 1$, where D_n has a standard exponential distribution, we obtain from this:

$$|\log Y_n| = O_p(1), \quad n \rightarrow \infty. \quad (2.7)$$

We now get from (2.4):

$$\frac{\log a_n - 3n}{\sqrt{5n}} = \frac{-2 \log V_{n-1} + \log Y_n - 3n}{\sqrt{5n}} = \frac{-2 \log V_{n-1} - 3n}{\sqrt{5n}} + O_p(n^{-1/2}),$$

as $n \rightarrow \infty$. Moreover, since

$$-2 \log V_{n-1} = -2 \sum_{i=1}^{n-1} \log \left(\frac{V_i}{V_{i-1}} \right) - 2 \log V_0 = -2 \sum_{i=1}^{n-1} \log \tau_i - 2 \log V_0, \quad (2.8)$$

we get by the central limit theorem:

$$\frac{\log a_n - 3n}{\sqrt{5n}} = \frac{-2 \sum_{i=1}^{n-1} \log \tau_i - 3n}{\sqrt{5n}} + o_p(1) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \quad n \rightarrow \infty, \quad (2.9)$$

where $\mathcal{N}(0, 1)$ denotes the standard normal distribution.

Let

$$B_1(m) = \sum_{i=1}^{[EN(1, m)]} (D_i - 1) / \sqrt{EN(1, m)},$$

and

$$B_2(m) = \{N(1, m) - EN(1, m)\} / \sqrt{\frac{5}{27} \log m},$$

and let, for fixed $y \in \mathbb{R}$, $n = n_{m, y} \in \mathbb{N}$ be defined by:

$$n = \left\lceil EN(1, m) + y \sqrt{\frac{5}{27} \log m} \right\rceil, \quad m \rightarrow \infty. \quad (2.10)$$

Then we find, using (2.5) and (2.9), as $m \rightarrow \infty$,

$$\begin{aligned} \mathbb{P}\{B_1(m) \geq x, B_2(m) \geq y\} &= \mathbb{P}\left\{B_1(m) \geq x, N(1, m) \geq EN(1, m) + y \sqrt{\frac{5}{27} \log m}\right\} \\ &\sim \mathbb{P}\{B_1(m) \geq x, N(1, m) \geq n\} = \mathbb{P}\{B_1(m) \geq x, \log a_n \leq \log m\} \\ &= \mathbb{P}\left\{B_1(m) \geq x, \frac{\log a_n - 3n}{\sqrt{5n}} \leq \frac{\log m - 3n}{\sqrt{5n}}\right\} \\ &\sim \mathbb{P}\left\{B_1(m) \geq x, \frac{-2 \sum_{i=1}^{n-1} \log \tau_i - 3n}{\sqrt{5n}} \leq \frac{\log m - 3EN(1, m) - y \sqrt{\frac{5}{3} \log m}}{\sqrt{5n}}\right\} \\ &\sim \mathbb{P}\{B_1(m) \geq x\} \mathbb{P}\left\{\frac{-2 \sum_{i=1}^{n-1} \log \tau_i - 3n}{\sqrt{5n}} \leq -\frac{y \sqrt{\frac{5}{3} \log m}}{\sqrt{\frac{5}{3} \log m}}\right\} \\ &= \mathbb{P}\{B_1(m) \geq x\} \mathbb{P}\left\{\frac{-2 \sum_{i=1}^{n-1} \log \tau_i - 3n}{\sqrt{5n}} \leq -y\right\}. \end{aligned}$$

where we use part (i), (2.10) and Corollary 1 (independence of the τ_i and the D_i) in the next to last line. Since, by (2.9),

$$\mathbb{P}\left\{\frac{-2 \sum_{i=1}^{n-1} \log \tau_i - 3n}{\sqrt{5n}} \leq -y\right\} \rightarrow \Phi(-y) = 1 - \Phi(y),$$

where Φ is the standard normal distributon function, the result now follows.

3. The central limit theorem

In this section we prove a 2-dimensional central limit theorem, by combining the results of the preceding section with the results in Groeneboom (1988).

Theorem 3.1. *Let $N(a, b)$ be the number of jumps in the interval $[a, b]$ of the process W , as defined in Definition 1, and let $D(a, b)$ be the area of the union of the triangles T_i , corresponding to points of jump $a_i \in [a, b]$, as defined in Theorem 2.1. Then:*

$$\left(\frac{5}{27} \log(b/a)\right)^{-1/2} \left(N(a, b) - \frac{1}{3} \log(b/a), D(a, b) - \frac{1}{3} \log(b/a)\right) \xrightarrow{\mathcal{D}} N(0, \Sigma), \quad b/a \rightarrow \infty,$$

where $N(0, \Sigma)$ is a bivariate normal distribution with expectation 0 and covariance matrix defined by

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & \frac{14}{5} \end{pmatrix} \quad (3.1)$$

Proof. As shown by the transformation to a stationary process (2.27) in Groeneboom (1988), the distribution of $N(a, b)$ only depends on the ratio b/a . The same construction shows that the distribution of $D(a, b)$ only depends on the ratio b/a . So we only have to prove the result for $a = 1$ and $b > 1$.

We know from Theorem 2.4 in Groeneboom (1988) that $EN(1, a) = \frac{1}{3} \log a$ and $\text{var}(N(1, a)) \sim (5/27) \log a$, as $a \rightarrow \infty$. Moreover,

$$D(1, a) = \sum_{a_i \in [1, a]} D_i = \sum_{a_i \in [1, a]} \text{area}(T_i),$$

where the T_i are the triangles of Theorem 2.1. So we can consider $D(1, a)$ as a random sum of standard exponential random variables, where the number of terms in the sum is equal to the random variable $N(a, b)$. Reasoning heuristically, as in the case of a compound Poisson distribution, we would get

$$E(D(1, a)) = EN(1, a) = \frac{1}{3} \log a,$$

and

$$\text{var}(D(1, a)) = EN(1, a) + \text{var}(N(1, a)) \sim \frac{1}{3} \log a + \frac{5}{27} \log a = \frac{14}{27} \log a.$$

We now show that we can prove the result by using this heuristic idea.

We write $D(1, a) - \frac{1}{3} \log a$ as the sum of the terms $A_1(a)$ and $A_2(a)$, where

$$A_1(a) = \sum_{i=1}^{[EN(1, a)]} D_i - \frac{1}{3} \log a,$$

defining $[EN(1, a)]$ as the largest integer not exceeding $EN(1, a) = \frac{1}{3} \log a$, and

$$A_2(a) = \begin{cases} \sum_{i=[EN(1, a)]+1}^{N(1, a)} D_i, & \text{if } N(1, a) > [EN(1, a)] \\ -\sum_{i=N(1, a)+1}^{[EN(1, a)]} D_i, & \text{if } N(1, a) \leq [EN(1, a)]. \end{cases}$$

We now have, if $N(1, a) > [EN(1, a)]$

$$\sum_{i=[EN(1, a)]+1}^{N(1, a)} D_i = \sum_{i=[EN(1, a)]+1}^{N(1, a)} (D_i - 1) + N(1, a) - [EN(1, a)],$$

and similarly, if $N(1, a) \leq [EN(1, a)]$,

$$-\sum_{i=N(1, a)+1}^{[EN(1, a)]} D_i = -\sum_{i=N(1, a)+1}^{[EN(1, a)]} (D_i - 1) + N(1, a) - [EN(1, a)],$$

where both sides are zero if $N(1, a) = [EN(1, a)]$. Hence we can write:

$$D(1, a) - \frac{1}{3} \log a = A_1(a) + N(1, a) - [EN(1, a)] + R(a),$$

where

$$R(a) = \begin{cases} \sum_{i=[EN(1, a)]+1}^{N(1, a)} (D_i - 1), & \text{if } N(1, a) > [EN(1, a)] \\ -\sum_{i=N(1, a)+1}^{[EN(1, a)]} (D_i - 1), & \text{if } N(1, a) \leq [EN(1, a)]. \end{cases}$$

Fix $\varepsilon > 0$. By Theorem 2.4 in Groeneboom (1988) there exists an $M = M(\varepsilon) > 0$ and an $a_0 = a_0(M)$ so that

$$\mathbb{P} \left\{ \left| \frac{N(1, a) - [EN(1, a)]}{\sqrt{\log a}} \right| > M \right\} < \varepsilon, \quad a \geq a_0.$$

Define

$$n_-(a) = [EN(1, a)] - M\sqrt{\log a}, \quad n_+(a) = [EN(1, a)] + M\sqrt{\log a}.$$

Then, by Doob's inequality,

$$\begin{aligned} & \mathbb{P} \left\{ \max_{m \in [[EN(1, a)]+1, n_+(a)]} \left| \sum_{i=[EN(1, a)]}^m (D_i - 1) \right| > \varepsilon \sqrt{\log a} \right\} \\ & + \mathbb{P} \left\{ \max_{m \in [n_-(a), [EN(1, a)]]} \left| \sum_{i=m}^{[EN(1, a)]} (D_i - 1) \right| > \varepsilon \sqrt{\log a} \right\} \\ & \leq \frac{n_+(a) - n_-(a) + 1}{\varepsilon^2 (\log a)} \sim \frac{2M}{\varepsilon^2 \sqrt{\log a}} \rightarrow 0, \quad a \rightarrow \infty. \end{aligned}$$

These relations imply: $R(a)/\sqrt{\log a} = o_p(1)$, $a \rightarrow \infty$, and hence:

$$\frac{D(1, a) - EN(1, a)}{\sqrt{\log a}} = \frac{\sum_{i=1}^{[EN(1, a)]} (D_i - 1)}{\sqrt{\log a}} + \frac{N(1, a) - [EN(1, a)]}{\sqrt{\log a}} + o_p(1). \quad (3.2)$$

The result now follows from Corollary 2 and Theorem 2.4 in Groeneboom (1988).

Using the methods from Groeneboom (1988) in going from the Poisson approximation to the sample process, one can now easily deduce the central limit result Theorem 1.1 from Theorem 3.1. The latter method is also used in Nagaev and Khamdamov (1991).

Remark 3.1. Instead of working directly with relation (2.2), expressing the differences between successive slopes of the convex hull in terms of the area of the corresponding rectangle and the y -coordinate of vertex at the intersection of the line segments with these slopes, Nagaev and Khamdamov (1991) write this relation first in the following form:

$$D_i = \frac{1}{2}V(a_{i-1})^2 \left(\frac{U(a_i) - U(a_{i-1})}{V(a_{i-1}) - V(a_i)} - \frac{U(a_{i-1}) - U(a_{i-2})}{V(a_{i-2}) - V(a_{i-1})} \right), \quad (3.3)$$

and then deduce a recursive relation for the $U(a_i)$ in terms of the $V(a_i)$ and D_i from this. They then define the random time

$$\theta_T = \inf \{i : U(a_i) \geq T\},$$

and consider sums of the form $\sum_{i=1}^{\theta_T} D_i$. This seems to lead to more complicated proofs.

Remark 3.2. The scaling constants for the central limit theorem for the area in Cabo and Groeneboom (1994) are not correct, although a correct application of the methods used in that paper would lead to the central limit theorem for the area, which is part of the central limit theorem 1.1 above. We here tried to present the results of the unpublished preprint Nagaev and Khamdamov (1991) in an easily understandable way, where the presentation is considerably simplified by the use of martingales, Doob's inequality and the results from Groeneboom (1988). In view of this simpler approach, and also the fact that Theorem 1.1 is in fact a stronger (2-dimensional) result, this approach seems preferable to the approach in Cabo and Groeneboom (1994). On the other hand, the computations along the lines of Cabo and Groeneboom (1994) give precise information on the first and second moments, as shown below in section 4.

Although Nagaev (1995) hints at the proof of the central limit theorem 1.1, there are many important missing steps, which have to be traced down to the unpublished preprint Nagaev and Khamdamov (1991). It seems fair to say that without knowledge of this preprint, deducing the result from Nagaev (1995) is pretty hard. Moreover, Nagaev (1995) contains in the crucial relation (3.7) an incorrect scaling constant (the constant $5/4$ there should be $20/27$), which further complicates the derivation of Theorem 1.1. For this reason we gave a simplified and self-contained treatment above.

Remark 3.3. Buchta (2005) gives the following relation between the sample variances of N_n and \bar{A}_n (using the notation of Theorem 1.1):

$$\frac{(n+1)(n+2)\text{var}(\bar{A}_n)}{n^2} = \text{var}(N_n) + d_{n+2},$$

where

$$d_n = (EN_n)^2 - \frac{n(EN_{n-1})^2}{n-1} - (2n-1)EN_n + 2nEN_{n-1} \sim EN_n \sim \frac{9}{5}\text{var}(N_n), \quad n \rightarrow \infty.$$

Hence

$$\text{var}(\bar{A}_n) \sim \frac{14}{5} \text{var}(N_n), \quad n \rightarrow \infty,$$

in accordance with the covariance matrix Σ in Theorem 1 in Nagaev and Khamdamov (1991) (Theorem 1.1 above). Note that the split-up of the variance of \bar{A}_n corresponds to the split-up (3.2) above, where d_{n+2} corresponds to the variance of the exponentials ξ_i in (3.2) and $\text{var}(N_n)$ corresponds to the variance of the second term on the right-hand side of (3.2).

Theorem 2 of Buchta (2003) gives for the number of vertices N_n of the convex hull of the points $(0, 1)$, $(1, 0)$ and P_1, \dots, P_n , where P_1, \dots, P_n is a uniform sample from the interior of the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$:

$$EN_n = \frac{1}{3} \left\{ 2 \sum_{i=1}^n \frac{1}{i} + 1 \right\},$$

and

$$\text{var}(N_n) = \frac{1}{27} \left\{ 10 \sum_{i=1}^n \frac{1}{i} + 12 \sum_{i=1}^n \frac{1}{i^2} - 28 + \frac{12}{n+1} \right\}.$$

This gives:

$$EN_n \sim \frac{2}{3} \log n, \quad \text{var}(N_n) \sim \frac{10}{27} \log n, \quad n \rightarrow \infty, \quad (3.4)$$

which corresponds to the distribution results derived in Groeneboom (1988), as is also noted in Buchta (2003).

The results in Groeneboom (1988) and Nagaev and Khamdamov (1991) only imply that one gets a normal limit distribution for the number of vertices of the convex hull of a uniform sample from the interior of a convex polygon with r vertices by centering with $\frac{2}{3}r \log n$ and dividing by $(\frac{10}{27}r \log n)^{1/2}$. It is not proved there that the variance of the number of vertices itself is also of order $\frac{10}{27}r \log n$. In principle one could have a central limit theorem where the scaling needed to get the central limit result is different from what one gets from the actual variance.

However, the only thing that still seems needed to go from (3.4) to the result that the variance itself is also of order $\frac{10}{27}r \log n$ seems the appropriate use of the independence of what happens in the corners of the polygons, so that one can conclude that the variance is the sum of the variances of the number of vertices in these corners. Moreover, one has to go from what happens in the triangle to what happens in the corners of the polygon. This is the subject of current research by Buchta. Results for higher moments of the convex hull of a uniform sample from triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ are given in Buchta (2011).

4. Simulations

Let $N(a, b)$ and $D(a, b)$ be defined as in Theorem 3.1. The distribution of these random variables only depends on the ratio b/a and in this section we present some simulation results for these random variables, taking $a = 1$ and replacing b by a .

The algorithm, given in section 4 of Nagaev (1995), was used to simulate part of the boundary of the convex hull of a Poisson process with intensity 1 in the first quadrant. The starting triangle is bounded by the x -axis, y -axis and a line of the form $x + y = c$, where $c > 0$. Its area D_0 has a standard exponential distribution and the point $W(1)$ is uniformly distributed on the line segment which is the hypotenuse of this triangle.

With the algorithm of Nagaev (1995) we can now generate the points $W(a)$, $a \geq 1$, and simulate in this way the distribution of $N(1, a)$ and $D(1, a)$. We start with $N(1, a)$ and recall the exact expressions for the expectation $EN(1, a)$ and $\text{var}(N(1, a))$ from Groeneboom (1988), Theorem 2.4:

$$EN(1, a) = \frac{1}{3} \log a, \quad (4.1)$$

and

$$\text{var}(N(1, a)) = \frac{5}{27} \log a + \frac{4}{9} (\tan^{-1}(\sqrt{a-1}))^2 + \frac{8}{9} \left\{ \frac{\tan^{-1}(\sqrt{a-1})}{\sqrt{a-1}} - 1 \right\}. \quad (4.2)$$

As noted on top of page 34 in Cabo and Groeneboom (1994), the formula for the variance of $(N(1, a))$, given in Theorem 2.1 of Groeneboom (1988) contained a typo (the argument of the first \tan^{-1} above was a instead of $\sqrt{a-1}$), and the correct formula is in fact given on p. 365 of Groeneboom (1988) (which we use here). Note that these are exact expressions for $EN(1, a)$ and $\text{var}(N(1, a))$ and not asymptotic ones.

The following table shows the means and variances for 10,000 simulations for the values $\log a = 10, 50$ and 100 . The exact values are given in 4 decimals accuracy.

Table 1. Comparison of $EN(1, a)$ and $\text{Var}(N(1, a))$ with simulated and asymptotic values.

$\log a$	simulated $EN(1, a)$	exact $EN(1, a)$	simulated $\text{Var}(N(1, a))$	exact $\text{Var}(N(1, a))$	asymptotic $\text{Var}(N(1, a))$
10	3.3519	3.3333	2.1193	2.0596	1.8519
50	16.6668	16.6667	9.5908	9.4670	9.2593
100	33.4259	33.3333	18.7039	18.7263	18.5185

It is seen from Table 1 that $EN(1, a)$ and $\text{Var}(N(1, a))$ are quite close to the simulated values and that, not unexpectedly, for $a = 10$ the exact expression for the variance of $N(1, a)$, given by (4.2), is closer to the simulated value than the asymptotic value.

We similarly did 10,000 simulations for the values $\log a = 10, 50$ and 100 to simulate the behavior of $D(1, a)$. Using the (corrected) methods of computation of Cabo and Groeneboom (1994) (details are given in Groeneboom (2011b)), it can be shown that

$$ED(1, a) = \frac{1}{3} \log a,$$

and, defining $\alpha = a - 1$, that:

$$\begin{aligned} \text{var}(D(1, a)) \\ = \frac{14}{27} \log a + \frac{2}{3\alpha^2} + \frac{4}{9\alpha} - \frac{44}{45} - \frac{2\{3 + \alpha(3 - 4\alpha)\} \tan^{-1}(\sqrt{\alpha})}{9\alpha^{5/2}} + \frac{4}{9} (\tan^{-1}(\sqrt{\alpha}))^2. \end{aligned}$$

These are again exact expressions for $ED(1, a)$ and $\text{var}(D(1, a))$ and not asymptotic ones. We get the following results.

Table 2. Comparison of $ED(1, a)$ and $\text{Var}(D(1, a))$ with simulated and asymptotic values.

$\log a$	simulated $ED(1, a)$	exact $ED(1, a)$	simulated $\text{Var}(D(1, a))$	exact $\text{Var}(D(1, a))$	asymptotic $\text{Var}(D(1, a))$
10	3.3664	3.3333	5.4089	5.3040	5.1852
50	16.6576	16.6667	26.1452	26.0448	25.9259
100	33.4933	33.3333	52.3304	51.9707	51.8519

We finally turn our attention to relation (3.7) in Nagaev (1995). This relation gives asymptotic expressions for the expectation and variance of the number ν_t of vertices falling in a disk S_t with radius t and center $(0, 0)$. On the basis of the results in Groeneboom (1988), it is to be expected that

$$E\nu_t \sim \frac{4}{3} \log t, \quad \text{var}(\nu_t) \sim \frac{20}{27} \log t, \quad t \rightarrow \infty, \quad (4.3)$$

whereas relation (3.7) in Nagaev (1995) gives the above relation for $E\nu_t$, but $(5/4) \log t$ as the asymptotic expression for $\text{var}(\nu_t)$. The argument for (4.3) is that, first of all, ν_t can be expected to behave asymptotically as the number of vertices with coordinates $x > y$ such that $x < t$ plus the number of vertices with coordinates $y \geq x$ such that $y < t$, since vertices with large x -coordinates will with high probability be very close to the x -axis and vertices with large y -coordinates will with high probability be very close to the y -axis. Secondly, again by Groeneboom (1988), the number of vertices with coordinates $x > y$ such that $x < t$ will behave asymptotically as $N(1, t^2)$, and similarly, the number of vertices with coordinates $y \geq x$ such that $y < t$ will behave asymptotically as $N(1/t^2, 1)$.

By the construction of the algorithm in Nagaev (1995), we can simulate the number of vertices $W(a)$, $a \geq 1$, satisfying $U(a)^2 + V(a)^2 < t^2$, by running the algorithm till we get a vertex $W(a)$ such that

$$U(a)^2 + V(a)^2 \geq t^2.$$

The resulting asymptotic behavior of $E\nu_t$ and $\text{Var}(\nu_t)$ is obtained from this by multiplying the results by the factor 2. The table below shows the result for 10,000 simulations for the values $\log t = 10, 50$ and 100.

Table 3. Comparison of $E\nu_t$ and $\text{Var}(\nu_t)$ with simulated and asymptotic values.

$\log t$	simulated $E\nu_t$	exact $E\nu_t$	simulated $\text{Var}(\nu_t)$	$(20/27) \log t$	$(5/4) \log t$
10	13.0778	13.3333	7.2630	7.40741	12.5
50	66.4792	66.6667	37.6192	37.0370	62.5
100	133.1330	133.3333	74.542	74.0741	125

Table 3 clearly suggests that the factor $5/4$ is much too large and that the correct approximation is indeed given by (4.3) above.

5. Concluding remarks

There is a remarkable analogy between the behavior of the left-lower convex hull of the Poisson point process, discussed above, and the least concave majorant of (one-sided) Brownian motion without drift, as analyzed in Groeneboom (1983). In the same way there is an analogy between the behavior of the lower convex hull of the Poisson point process inside a parabola, as analyzed in Groeneboom (1988) and Nagaev (1995), and the least concave majorant of Brownian motion with a parabolic drift, as studied in Groeneboom (1989) and Groeneboom (2011a). Why this is the case is still somewhat of a mystery and deserves (in my view) further investigation.

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